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# A NEW APPROACH FOR LINEAR STABILITY ANALYSIS

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ABSTRACT

This paper investigates the issue of linear stability analysis for two and three level explicit and implicit one-dimensional finite different numerical schemes. A new approach which is based on the von Neumann method is presented. This approach was validated by testing some popular numerical schemes.

KEY WORDS Linear stability analysis Finite difference Explicit and implicit schemes

## INTRODUCTION

As is well known, a numerical method is useless if the method will not converge to the differential equation. To prove convergence for nonlinear systems of equations is currently impossible for most cases. For the simpler case of a scalar equation, particularly the linear scalar equation, the analysis is possible. Although linear convergence is not a sufficient condition for guaranteeing nonlinear convergence, it is still a necessary condition to achieve nonlinear convergence. To prove convergence, there is a fundamental Lax equivalence theory[1] for linear finite difference methods, which declares that for a *consistent* linear method *stability* is necessary and sufficient for *convergence*. Here linear convergence is obtained by two sequential conditions: one, the numerical method has to be consistent with the PDE simulated; two, the numerical method has to be stable for, at least, smooth intitial data. The numerical methods concerned in the paper are consistent methods. Therefore, all that is left to prove convergence is proving that these methods are linearly stable.

At present, there are several techniques available to analyse linear stability. This includes the discrete perturbation method, the Hirt method, the matrix method and the von Neumann method[2,3,4]. Details of these methods can also be found in Reference 5. Comparing with other techniques the von Neumann method is the most widely applied technique. However, it is by no means an easy task, using these methods, to analyse linear stability even for one-dimensional, constant coefficient, initial value problems. For a numerical scheme generated from a complex high-order PDE or for a numerical method which has more than second-order accuracy, the linear stability analysis can be extremely complicated when applying these techniques. Normally, quite tedious and complicated algebraic functions or matrices will be encountered, which are either very difficult to analyse or even impossible to manipulate. Often, numerical schemes cannot be applied because of lack of stability information. Obviously, a simple and reliable method for proving linear stability is desired.

In this paper we investigate this issue and develop an approach to the linear stability analysis in a simple manner.

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## LINEAR STABILITY ANALYSIS

In this section we restrict our study to the initial value problem (IVP) for the simplest case of one dimensional scalar linear PDEs with the smooth initial data.

We discretise the computational half plane by choosing a uniform mesh with a cell width $h=\Delta x$, a time step $k=\Delta t$ and define the computational grid $x_j=jh$, $t_n=nk$. We use $U_j^n$ to denote the computed approximation to the exact solution $u(x_j, t_n)$ of the PDEs.

### *A new approach for linear stability analysis*

For one dimensional linear finite difference numerical methods with smooth initial data,

$$\sum_{k^{n+1}} B_{k^{n+1}}^{n+1} U_{j+k^{n+1}}^{n+1} = \sum_{k^n} B_{k^n}^n U_{j+k^n}^n + \sum_{k^{n-1}} B_{k^{n-1}}^{n-1} U_{j+k^{n-1}}^{n-1} \tag{1}$$

if the amplification coefficient $|\lambda(\theta)|$ of the scheme is a monotone (increases or decreases) function, i.e. $(\lambda(\theta)\overline{\lambda(\theta)})' \geqslant 0$ (or $\leqslant 0$), with respect to $\theta$ in the interval $[0, \pi]$, then the linear stability conditions of the scheme can be determined from,

$$\lambda = \frac{\sum_{k^n}(-1)^{|k^n|} B_{k^n}^n + \sum_{k^{n-1}}(-1)^{|k^{n-1}|} \frac{1}{\lambda} B_{k^{n-1}}^{n-1}}{\sum_{k^{n+1}}(-1)^{|k^{n+1}|} B_{k^{n+1}}^{n+1}} \tag{2}$$

$$\lambda = \frac{-\sum_{k^{n-1}} B_{k^{n-1}}^{n-1}}{\sum_{k^n} B_{k^n}^n + \sum_{k^{n-1}} B_{k^{n-1}}^{n-1}} \tag{3}$$

For pure odd grid point or pure even grid point finite difference numerical schemes if the amplification coefficient is a concave or convex function, i.e. $|\lambda(\theta)|'' \geqslant 0$ (or $\leqslant 0$), in the interval $[0, \pi]$, then an additional stability condition of the scheme is required,

$$\lambda = \frac{\sum_{k^n} \operatorname{sign}^n B_{k^n}^n + \sum_{k^{n-1}} \operatorname{sign}^{n-1} \frac{1}{\lambda} B_{k^{n-1}}^{n-1}}{\sum_{k^{n+1}} \operatorname{sign}^{n+1} B_{k^{n+1}}^{n+1}} \tag{4}$$

here,

$$\begin{cases} \operatorname{sign}^n = \sin \dfrac{k^n \pi}{2} & \forall \text{ odd number } k^n \\ \operatorname{sign}^n = \cos \dfrac{k^n \pi}{2} & \forall \text{ even number } k^n \end{cases} \tag{5}$$

where $k^n$ are the integer grid point numbers at time level $n$; $B_{k^n}^n$ are constant coefficients; $\lambda(\theta)$ is the amplification factor of the numerical scheme; $\overline{\lambda(\theta)}$ is the conjugate of the $\lambda(\theta)$. To achieve stability,

$$|\lambda| \leqslant 1 \tag{6}$$

If the amplification coefficient does not satisfy the conditions above, then the stability conditions can be defined by investigation of those phase angles at which the amplification coefficient has extreme values.

*Proof*

The von Neumann method (Fourier Series method) is based on assuming that,

$$U_j^n = A_L^n e^{iLj\Delta x} \tag{7}$$

where $A_L^n$ is the amplitude at time level $n$; $L$ is the wave number in $x$-direction, $L=2\pi/\tau$, $\tau$ is the wavelength; $i$ is the complex number, $i=\sqrt{-1}$.

Considering the general form of linear numerical methods of (1), from (7) we have,

$$\begin{cases} U_{j+k^{n+1}}^{n+1}=A_L^{n+1}\,e^{iL(j+k^{n+1})\Delta x} \\ U_{j+k^n}^{n}=A_L^{n}\,e^{iL(j+k^{n})\Delta x} \\ U_{j+k^{n-1}}^{n-1}=A_L^{n-1}\,e^{iL(j+k^{n-1})\Delta x} \end{cases} \tag{8}$$

Substituting (8) into (1),

$$\sum_{k^{n+1}} B_{k^{n+1}}^{n+1}A_L^{n+1}\,e^{iL(j+k^{n+1})\Delta x}=\sum_{k^n} B_{k^n}^{n}A_L^{n}\,e^{iL(j+k^{n})\Delta x}+\sum_{k^{n-1}} B_{k^{n-1}}^{n-1}A_L^{n-1}\,e^{iL(j+k^{n-1})\Delta x} \tag{9}$$

Dividing both sides of (9) by $A_L^n\,e^{iLj\Delta x}$ and reorganizing it, we get the amplification factor at the new time level,

$$\begin{aligned}\lambda^n(\theta)&=\frac{A_L^{n+1}}{A_L^n}\\ &=\frac{\sum_k\left(B_{k^n}^n e^{ik^n\theta}+\dfrac{1}{\lambda^{n-1}}B_{k^{n-1}}^{n-1}\,e^{ik^{n-1}\theta}\right)}{\sum_{k^{n+1}}B_{k^{n+1}}^{n+1}\,e^{ik^{n+1}\theta}}\\ &=\gamma_r+i\gamma_i\end{aligned} \tag{10}$$

Here $\theta$ is the phase angle, $\theta=L\Delta x$;

$$\gamma_r=\frac{\left(\sum_k b_1+\dfrac{1}{\lambda^{n-1}}\sum_k c_1\right)\sum_k a_1+\left(\sum_k b_2+\dfrac{1}{\lambda^{n-1}}\sum_k c_2\right)\sum_k a_2}{(\sum_k a_1)^2+(\sum_k a_2)^2} \tag{11}$$

$$\gamma_i=\frac{\left(\sum_k b_2+\dfrac{1}{\lambda^{n-1}}\sum_k c_2\right)\sum_k a_1-\left(\sum_k b_1+\dfrac{1}{\lambda^{n-1}}\sum_k c_1\right)\sum_k a_2}{(\sum_k a_1)^2+(\sum_k a_2)^2} \tag{12}$$

where,

$$\begin{aligned} a_1&=B_k^{n+1}\cos k^{n+1}\theta\\ a_2&=B_k^{n+1}\sin k^{n+1}\theta\\ b_1&=B_k^{n}\cos k^{n}\theta\\ b_2&=B_k^{n}\sin k^{n}\theta\\ c_1&=B_k^{n-1}\cos k^{n-1}\theta\\ c_2&=B_k^{n-1}\sin k^{n-1}\theta \end{aligned}$$

For a linear numerical method with smooth initial data, since the Courant number is constant, therefore the amplification factors at different time levels are identical, i.e. $\lambda^n=\lambda^{n-1}$. From now on we write $\lambda$ instead for simplicity.

The absolute value of the amplification factor $|\lambda|$ is called amplifier coefficient. Obviously if $|\lambda|>1$, the numerical method will not be stable, otherwise, it is stable. Therefore, for stability,

$$|\lambda|=\sqrt{\gamma_r^2+\gamma_i^2}\leqslant 1 \tag{13}$$

for all phase angles ranging from $\theta=0$ to $\theta=\pi$.

This is the normal approach of analysing the stability in practice using the Fourier method. But, generally, (13) is very complicated algebrically, especially for high order numerical methods, say, over second order. For a method over second order, $|\lambda|$ is very difficult to work out, or even impossible to manipulate. Here, we are going to adopt a new approach.

The difficulty of analysing (13) lies in the phase angle $\theta$ which covers the whole domain from 0 to $\pi$ associated with all wave numbers. The question here is that as far as the stability of a numerical scheme is concerned, is it necessary to analyse the whole range of the phase angles? If not, which phase angle do we need to analyse? The instability of a numerical method is caused by the unbounded fast accumulated amplitude error with the time evolution. To limit the amplitude error we need first to find out at which phase angles the amplification coefficient have the extreme values in the interval $[0, \pi]$ (we called the angles which represent all turning point angles and boundary point angles extreme value angles), and then it is sufficient to restrict these values to less than or equal to 1 at these phase angles. In order to find the angles at which the $|\lambda|$ has extreme values, first we need the first derivative of $|\lambda|$ with respect to $\theta$, i.e. $|\lambda|'$, then by setting $|\lambda|'$ equals to zero the extreme value angles can be defined.

From (13) we have,

$$|\lambda|' = \frac{\gamma_r\gamma_r' + \gamma_i\gamma_i'}{\sqrt{\gamma_r^2 + \gamma_i^2}} = 0 \tag{14}$$

Equation (14) is equivalent to

$$(\gamma_r^2 + \gamma_i^2)' = 0 \tag{15}$$

or

$$(\lambda(\theta)\overline{\lambda(\theta)})' = 0 \tag{16}$$

Here,

$$\gamma_r^2 + \gamma_i^2 = \frac{\left(\sum_k b_1 + \frac{1}{\lambda}\sum_k c_1\right)^2 + \left(\sum_k b_2 + \frac{1}{\lambda}\sum_k c_2\right)^2}{(\sum_k a_1)^2 + (\sum_k a_2)^2} \tag{17}$$

therefore,

$$\begin{aligned}(\gamma_r^2 + \gamma_i^2)' = {} & 2[(\textstyle\sum a_1)^2 + (\sum a_2)^2]\Big[\left(\sum b_2 + \frac{1}{\lambda}\sum c_2\right)\left(\sum k^n b_1 + \frac{1}{\lambda}\sum k^{n-1} c_1\right) \\ & -\left(\textstyle\sum b_1 + \frac{1}{\lambda}\sum c_1\right) + \left(\sum k^n b_2 - \frac{1}{\lambda}\sum k^{n-1} c_2\right)\Big] \\ & -2\Big[\left(\textstyle\sum b_1 + \frac{1}{\lambda}\sum c_1\right)^2 + \left(\sum b_2 + \frac{1}{\lambda}\sum c_2\right)^2\Big] \\ & \times [\textstyle\sum a_2 \sum k^{n+1} a_1 - \sum a_1 \sum k^{n+1} a_2]/[(\sum a_1)^2 + (\sum a_2)^2]^2\end{aligned} \tag{18}$$

For 3-level explicit schemes (18) reduced to,

$$(\gamma_r^2 + \gamma_i^2)' = 2\Big[\left(\textstyle\sum b_2 + \frac{1}{\lambda}\sum c_2\right)\left(\sum k^n b_1 + \frac{1}{\lambda}\sum k^{n-1} c_1\right) - \left(\sum b_1 + \frac{1}{\lambda}\sum c_1\right)\left(\sum k^n b_2 - \frac{1}{\lambda}\sum k^{n-1} c_2\right)\Big] \tag{19}$$

For 2-level explicit schemes (18) is further reduced to,

$$(\gamma_r^2+\gamma_i^2)'=2(\sum b_2 \sum k^n b_1-\sum b_1 \sum k^n b_2) \tag{20}$$

By solving equation (15) the obvious Courant number-independent extreme value angles can be easily defined. They are

$$\theta_1=0 \tag{21}$$

$$\theta_2=\pi \tag{22}$$

$$\theta_3=\frac{\pi}{2} \quad \forall \textit{ either odd or even } k \tag{23}$$

since when $\theta=0$, $\pi$, $a_2$, $b_2$, and $c_2$ equal to zeros, therefore $(\gamma_r^2+\gamma_i^2)'=0$; when $\theta=\pi/2$ $a_1$, $b_1$, $c_1$ are zeros $\forall$ odd $k$ and $a_2$, $b_2$, $c_2$ are zeros $\forall$ even $k$, resulting in $(\gamma_r^2+\gamma_i^2)'=0$.

Equation (23) means that for pure odd or even number grid point schemes the amplification coefficient $|\lambda|$ has a extreme value at phase angle $\theta=\pi/2$. For example, the Lax-Friedrichs scheme which is a odd number point scheme has a extreme value angle at the angle $\theta=\pi/2$.

There may be other Courant number-dependent extreme value angles between $\theta=0$ and $\theta=\pi$ depending on the solution of (15). However, there is one important category of schemes for which the amplification coefficient is a monotone function, that means $(\lambda(\theta)\overline{\lambda(\theta)})'\geqslant 0$ (or $\leqslant 0$) $\forall$ $[0, \pi]$. In this case, the extreme value angles must be either at $\theta=0$ or at $\theta=\pi$, in which cases the linear stability analysis becomes very simple. Actually, as we will see later, large number of useful finite difference numerical schemes fall into this category.

For pure odd or even grid point schemes if $|\lambda(\theta)|''\geqslant 0$ (or $\leqslant 0$) $\forall$ $[0, \pi]$, i.e. the function curve of the amplification coefficient is either concave or convex, then the extreme value angle may appear at $\theta=\pi/2$. Hence we have the following criterion:

*Criterion* For finite different numerical schemes with smooth initial data it is necessary and sufficient to investigate the linear stability at phase angles at which $|\lambda(\theta)|$ has extreme values in the interval $[0, \pi]$.

If $(\lambda(\theta)\overline{\lambda(\theta)})'\geqslant 0$ (or $\leqslant 0$) in the interval $[0, \pi]$, it is necessary and sufficient to investigate the linear stability at the phase angle $\theta=0$ and $\theta=\pi$.

For pure odd or even grid point finite difference schemes if $|\lambda(\theta)|''\geqslant 0$ (or $\leqslant 0$), it is necessary and sufficient to investigate the linear stability at phase angle $\theta=0$, $\theta=\pi$ and $\theta=\pi/2$.

Based on this criterion substituting $\theta=0$, $\pi$ and $\theta=\pi/2$ into (10) we establish the new approach introduced at the beginning of the section.

Equations (2) and (3) are the general form of amplification function which is valid for two and three times levels, explicit and implicit numerical schemes. For convenience here we give some specific schemes as follows.

*3-level explicit schemes*

If we consider 3-level explicit schemes

$$U_j^{n+1}=\sum_{k^n} B_{k^n}^n U_{j+k^n}^n+\sum_{k^{n-1}} B_{k^{n-1}}^{n-1} U_{j+k^{n-1}}^{n-1} \tag{24}$$

then (2) and (3) become,

$$\lambda=\sum_{k^n}(-1)^{|k^n|}B_{k^n}^n+\frac{1}{\lambda}\sum_{k^{n-1}}(-1)^{|k^{n-1}|}B_{k^{n-1}}^{n-1} \tag{25}$$

$$\lambda=-\sum_{k^{n-1}} B_{k^{n-1}}^{n-1} \tag{26}$$

*2-level explicit schemes*

If we consider 2-level explicit schemes,

$$U_j^{n+1}=\sum_{k^n} B_{k^n}^n U_{j+k^n}^n \tag{27}$$

then, (2) is further simplified to,

$$\lambda=1-2\sum_{k^n=\pm1,\pm3,\dots} B_{k^n}^n \tag{28}$$

since $\sum_k{}^n B_{k^n}^n=1$ for consistency.

*2-level implicit schemes*

For 2-level implicit schemes,

$$\sum_{k^{n+1}} B_{k^{n+1}}^{n+1} U_{j+k^{n+1}}^{n+1}=\sum_{k^n} B_{k^n}^n U_{j+k^n}^n \tag{29}$$

the amplification factor of equation (2) becomes,

$$\lambda=\frac{\sum_{k^n}(-1)^{|k^n|}B_{k^n}^n}{\sum_{k^{n+1}}(-1)^{|k^{n+1}|}B_{k^{n+1}}^{n+1}} \tag{30}$$

*2-level fully implicit schemes*

For fully implicit schemes,

$$\sum_{k^{n+1}} B_{k^{n+1}}^{n+1} U_{k^{n+1}}^{n+1}=U_j^n \tag{31}$$

The amplification factor has the following simple form,

$$\lambda=\frac{1}{1-2\sum_{k^{n+1}=\pm1,\pm3,\dots} B_{k^{n+1}}^{n+1}} \tag{32}$$

since $\sum_{k^{n+1}} B_{k^{n+1}}^{n+1}=1$ for consistency.

*Procedures of the new approach*

The procedures of linear stability analysis using the new approach can be outlined as follows:

1. calculate the extreme value angles $\theta(c)$ using (15) with (18)–(20)
2. check whether or not the extreme value angles $\theta(c)$ function conform to the monotone function requirements
3. if satisfy the requirements then the stability conditions can be defined applying (24)–(32) (according to the scheme used)
4. if not. using (13) with (17), analyse stability conditions only at the extreme value angles $\theta(c)$ defined at stage one.

## EXAMPLES OF STABILITY ANALYSIS

In this section we use some numerical schemes some of which the stability conditions are well known to illustrate the procedures and test the stability approach.

*Example 1. Modified Lax-Wendroff scheme*

Consider the scheme,

$$U_j^{n+1}=U_j^n-\frac{c}{2}(U_{j+1}^n-U_{j-1}^n)+\frac{d}{2}(U_{j+1}^n-2U_j^n+U_{j-1}^n) \tag{33}$$

here, $c$ is the Courant number, $c=a\Delta t/\Delta x$, $a$ is the wave spped, $d$ is a variable.

From (20) we have,

$$(\gamma_r^2+\gamma_i^2)'=2[(c^2-d^2)\cos\theta-(1-d)d]\sin\theta \tag{34}$$

Two special cases are easily defined from (34): when $d=c^2$ and $d=|c|$ the amplification coefficient is monotone, since in these cases (34) keeps the same sign in the interval $[0,\pi]$. In the former case (33) becomes the second-order Lax-Wendroff scheme. From (28) the amplification factor is,

$$\lambda=1-2c^2 \tag{35}$$

Therefore the stability condition is,

$$|\lambda|\leqslant 1 \quad \text{for } |c|\leqslant 1 \tag{36}$$

This is identical to the familiar result. If the latter case (33) reduces to the first-order upwind scheme. The scheme is stable for $|c|\leqslant 1$.

For $d$ being other values the $|\lambda|$ is not always a monotone function. Its behaviour is determined by Courant number and $d$. In this case we need to find out the extreme value angle function $\theta(c)$ by setting (34) equal to 0, which gives,

$$\cos\theta=\frac{(1-d)d}{c^2-d^2} \tag{37}$$

Bringing (37) into (13) we have,

$$\begin{aligned}|\lambda|&=\sqrt{\frac{1}{4}(1+\cos\theta)^2+c^2\sin^2\theta}\\&=\sqrt{\frac{1}{4}\left(1+\frac{(1-d)d}{c^2-d^2}\right)^2+c^2\left[1-\left(\frac{1-d)d}{c^2-d^2}\right)^2\right]}\end{aligned} \tag{38}$$

For stability $|\lambda|\leqslant 1$.

*Example 2. Leapfrog scheme*

The leapfrog scheme for the scalar advection equation has the following form,

$$U_j^{n+1}=U_j^{n-1}-cU_{j+1}^n+cU_{j-1}^n \tag{39}$$

This is a 3-level explicit scheme. It is easy to prove that the scheme has extreme values at $\theta=0$, $\pi$, and $\pi/2$. Using (25) we have,

$$\lambda=\frac{1}{\lambda} \tag{40}$$

That is,

$$\lambda^2=1 \tag{41}$$

From (26) we have

$$\lambda=-1 \tag{42}$$

From (4),

$$\lambda=-c\pm\sqrt{1+c^2} \tag{43}$$

Equation (41) and (42) mean $|\lambda|=1$; (43) means $|\lambda|\leqslant 1$ for $|c|\geqslant 0$. Actually this scheme is neutrally stable for $|c|\leqslant 1$.

*Example 3. Crank-Nicolson scheme*

$$U_j^{n+1}+\frac{1}{4}cU_{j+1}^{n+1}-\frac{1}{4}cU_{j-1}^{n+1}=U_j^n-\frac{1}{4}cU_{j+1}^n+\frac{1}{4}cU_{j-1}^n \tag{44}$$

This is a 2-level implitic scheme. The scheme has a monotone amplification coefficient function. From (30),

$$\lambda=\frac{1+\frac{1}{4}c-\frac{1}{4}c}{1-\frac{1}{4}c+\frac{1}{4}c}=1 \tag{45}$$

i.e. $|\lambda|=1$. This scheme is unconditionally stable.

*Example 4. Lax-Friedrichs scheme*

$$U_j^{n+1}=\left(\frac{1}{2}-\frac{c}{2}\right)U_{j+1}^n+\left(\frac{1}{2}+\frac{c}{2}\right)U_{j-1}^n \tag{46}$$

This is a 2-level explicit odd point scheme. The scheme has a concave amplification coefficient function, therefore we need to check both (28) and (4). From (28),

$$|\lambda|=1 \tag{47}$$

From (4),

$$|\lambda|=|c| \tag{48}$$

Therefore, this scheme is stable if $|c|\leqslant 1$.

*Example 5. Fully discrete fourth-order scheme (see Reference 6)*

$$\begin{aligned}U_j^{n+1}=&\left(1+\frac{1}{4}c^4-\frac{5}{4}c^2\right)U_j^n+\left(\frac{1}{24}c^4+\frac{1}{12}c^3-\frac{1}{24}c^2-\frac{1}{12}c\right)U_{j-2}^n\\&+\left(\frac{2}{3}c+\frac{2}{3}c^2-\frac{1}{6}c^3-\frac{1}{6}c^4\right)U_{j-1}^n+\left(\frac{1}{6}c^3+\frac{2}{3}c^2-\frac{1}{6}c^4-\frac{2}{3}c\right)U_{j+1}^n\\&+\left(\frac{1}{12}c-\frac{1}{24}c^2-\frac{1}{12}c^3+\frac{1}{24}c^4\right)U_{j+2}^n\end{aligned} \tag{49}$$

This is a 2-level explicit fully discrete fourth-order both in space and time scheme. The scheme has a monotone amplification coefficient. From (28) the amplification factor is,

$$\lambda=1-\frac{8}{3}c^2+\frac{2}{3}c^4 \tag{50}$$

which is plotted in *Figure 1*. The stability conditions of the scheme are,

$$\begin{aligned}-2\leqslant c&\leqslant -1.73\\-1\leqslant c&\leqslant 1\\1.73\leqslant c&\leqslant 2\end{aligned} \tag{51}$$

*Example 6. Explicit space-centred scheme for the model diffusion equation $u_t=vu_{xx}$*

$$U_j^{n+1}=(1-2d)U_j^n+dU_{j+1}^n+dU_{j-1}^n \tag{52}$$

here, the $d$ is the diffusion number, $d=v\Delta t/(\Delta x)^2$; $v$ is the viscous coefficient.

It has been proved that the scheme has a monotone amplification coefficient function. From (28) the amplification factor,

$$\lambda=1-4d \tag{53}$$

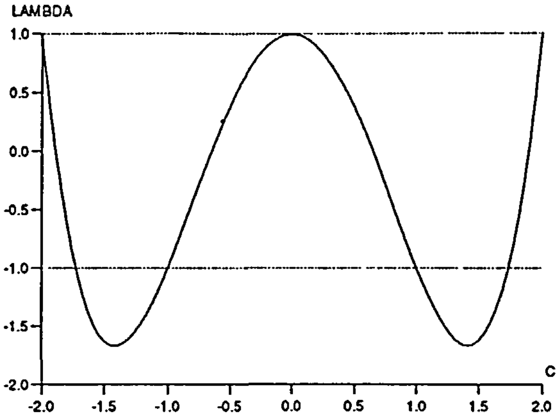


*Figure 1* Amplification factor of the 5-point scheme for model advection equation

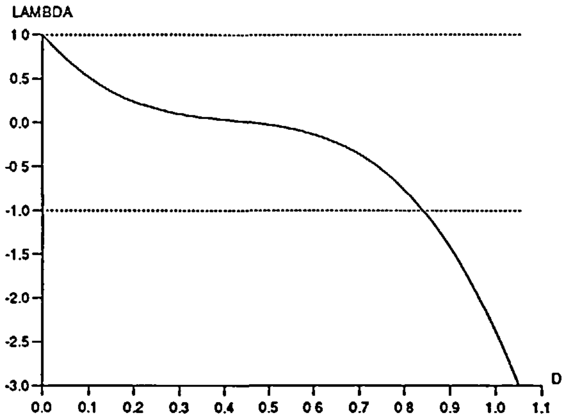


*Figure 2* Amplification factor of the 7-point scheme for model diffusion equation

The stable condition is,

$$|\lambda| \leqslant 1 \quad \text{for } d \leqslant \frac{1}{2} \tag{54}$$

*Example 7. Fully implicit scheme for the model diffusion equation*

Again the scheme has a monotone function

$$(1+2d)U_j^{n+1} - dU_{j+1}^{n+1} - dU_{j-1}^{n+1} = U_j^n \tag{55}$$

From (32),

$$\lambda = \frac{1}{1+4d} \tag{56}$$

Since $d$ is positive the scheme is unconditionally stable for $d>0$. We get the same conclusion as that proved by using other techniques.

*Example 8. Fully discrete seven point scheme for the model diffusion equation* (*see Reference 7*)

$$U_j^{n+1}=\left(1-\frac{10}{3}d^3+\frac{14}{3}d^2-\frac{49}{18}d\right)U_j^n+\left(\frac{15}{6}d^3-\frac{13}{4}d^2+\frac{3}{2}d\right)(U_{j-1}^n+U_{j+1}^n)$$
$$+\left(d^2-d^3-\frac{3}{20}d\right)(U_{j-2}^n+U_{j+2}^n)$$
$$+\left(\frac{1}{6}d^3-\frac{1}{12}d^2+\frac{1}{90}d\right)(U_{j-3}^n+U_{j+3}^n) \quad (57)$$

This is a fully discrete explicit scheme which has sixth-order accuracy in space and third-order in time. From (28) we have,

$$\lambda=1-\frac{32}{3}d^3+\frac{40}{3}d^2-\frac{272}{45}d \quad (58)$$

For $|\lambda|\leqslant 1$, see *Figure 2*, the stability condition is,

$$0\leqslant d\leqslant 0.85 \quad (59)$$

## CONCLUSIONS

In this paper we presented a linear stability analysis method for one dimensional numerical schemes. To illustrate the method linear stability of a variety of numerical schemes is analysed. This approach offers us a simple means to deal with linear stability study for 1-D finite difference schemes.

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